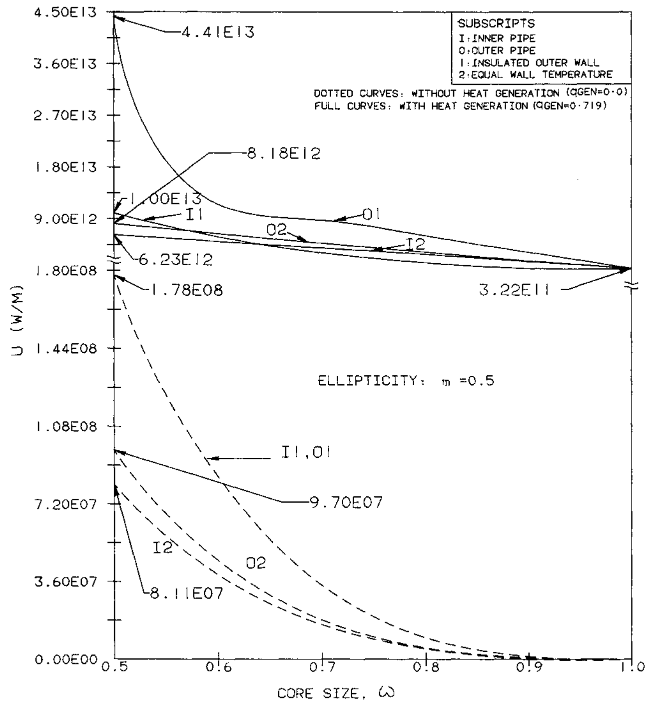

Fig. 1 Energy transfer vs core size for ellipticity of 0.5.

data given in Ref. 6. It is indeed easy to evaluate Eqs. (31) and (32) numerically for any other generation density. The intent is to show the effect of heat generation in the annulus on energy transfer through the walls of a confocal elliptical annulus.

Appendix

$$f_o'(1) = \frac{1}{4}\,[C_1 - 2(1-m^4)] \tag{A1}$$

$$f_o'(\omega) = \frac{1}{4}\left[\frac{C_1}{\omega} - 2\left(\omega - \frac{m^4}{\omega^3}\right)\right] \tag{A2}$$

$$e_o'(1) = \frac{1}{4}(1-m^8) - \frac{1-\omega^2}{1+\omega^2}\,m^4 - \frac{1}{4}\,C_1(1+m^4) - \frac{3}{16}\,C_1(1+\omega^2)\left(1+\frac{m^4}{\omega^2}\right) + \frac{1}{4}\,C_1^2\left(1+\frac{m^4}{\omega^2}\right)\Big/\left(1-\frac{m^4}{\omega^4}\right) \tag{A3}$$

where

$$\omega e_o'(\omega) = -\frac{1}{4}\,(1+m^4)\left[G - 2\left(\omega^2 - \frac{m^4}{\omega^2}\right)\right] + \frac{1-\omega^2}{1+\omega^2}\,m^4 + \frac{1}{2}\,C_1\left(\omega^2 - \frac{m^4}{\omega^2}\right)\log\omega - \frac{1}{4}\left[\left(\omega^2 - \frac{m^8}{\omega^4}\right) + \frac{1}{16}\,C_1(1+\omega^2)\left(1+\frac{m^4}{\omega^2}\right)\right] - \frac{1}{2}\,C_1\left(\omega^2 + \frac{m^4}{\omega^2}\right) + \frac{1}{4}\,C_1^2\left(1+\frac{m^4}{\omega^2}\right)\Big/\left(1-\frac{m^4}{\omega^2}\right) \tag{A4}$$

$$f_o'(1) - \omega f_o'(\omega) = -\frac{1}{2}\,(1-\omega^2)\left(1+\frac{m^4}{\omega^2}\right) \tag{A5}$$

$$e_o'(1) - \omega e_o'(\omega) = I_{00} \tag{A6}$$

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Computational Aspects of Heat Transfer in Structures via Transfinite-Element Formulation

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Introduction

THE computational aspects of transient thermal analysis of structures has been approached by several researchers[1,2] using finite elements; the most common being the step-by-step time-marching schemes, modal superposition techniques, etc. The present Note describes a viable computational approach with emphasis on a generalized transfinite-element methodology[3] for applicability for the computational aspects of heat transfer in structures. Highlights and characteristic features of the approach are described via general formulations and applications to sample one- and two-dimensional problems. Comparative numerical test cases therein validate the fundamental capabilities via the proposed concepts.

Transform-Methods-Based Finite-Element Methodology

Briefly summarized, the transform-methods-based finite-element approach combines the modeling versatility of contemporary finite elements in conjunction with transform methods and classical Galerkin schemes. In this Note, the formulations are applied for transient thermal analysis; however, the general concepts are applicable to other transient and/or interdisciplinary problems. First, the region under consideration is idealized as a finite number of discrete elements. Therein, numerical

Received Dec. 2, 1985; revision received Feb. 3, 1987.

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computations for each individual element in the transform domain yield element matrices that are assembled in the conventional manner and then solved in the transform domain itself. The actual solution is then obtained by employing an inverse numerical transform. In the proposed approach, Laplace transforms, Fourier transforms, or any appropriate transforms can be used in conjunction with finite elements. Herein, the finite-element formulations in the transform domain are referred to as "transfinite formulations," and the corresponding elements are referred to as "transfinite elements." In fact, the proposed approach may be used in conjunction with other numerical methods, such as finite-difference methods, boundary-element methods, etc. The characteristics features are as follows:

1) Use of an appropriate transform method.
2) Reduction in dimensionality of equations to facilitate general finite-element formulations in the transform domain.
3) Permits formulation of a large class of closed-form basis functions for transient one-dimensional problems, and allows general formulations for other cases.
4) Permits other numerical methods, such as finite differences, boundary-element methods, etc., in conjunction with the proposed methodology.
5) Eliminates the need to use step-by-step time-marching schemes and complex time-dependent algorithms, estimation of critical time steps, etc.
6) Permits the solution response at desired times of interest.

Thermal Analysis: Transfinite-Element Approach

General field problems can be solved by discretizing a given region into finite domains or elements. Consider the governing heat conduction equations for a three-dimensional solid (in domain R) bounded by a closed surface S given by

$$\rho c \frac{\partial T}{\partial t}(x_i,t) = -\frac{\partial q_i}{\partial x_i} + Q(x_i,t), \quad i = 1,2,3 \tag{1}$$

with boundary and initial conditions

$$T = T_s \text{ on } S_1 \tag{2a}$$

$$(\text{sum on } i) \qquad q_i n_i = -q_s \text{ on } S_2 \tag{2b}$$

$$(\text{sum on } i) \qquad q_i n_i = h(T - T_\infty) \text{ on } S_3 \tag{2c}$$

and

$$T(x_i,0) = T_i \tag{3}$$

where $S = S_1 + S_2 + S_3$; ρ is the density; c the specific heat; q_i, $i = 1,2,3$, are the components of heat flow rate per unit area; Q the internal heat generation rate per unit volume; T_s the specified temperature; q_s the surface heating per unit area; and T_∞ the convective medium temperature.

Basically, conventional heat-transfer formulations for Eqs. (1–3), when discretized by the finite-element technique, lead to

$$[C]\{\dot{T}\} + [K]\{T\} = \{R\} \tag{4}$$

The class of formulations for solving Eqs. (4) is termed in literature[1] as step-by-step time-marching schemes that are either implicit or explicit and require estimating critical time steps, etc.

Methodology and Formulations

Focusing attention on the transfinite-element methodology, after discretizing the physical domain of interest, and selecting the Laplace transform as the appropriate transform method and applying this to the governing heat conduction equations (1–3), the discretized equations (at the element level) invoking the classical Bubnov-Galerkin scheme can be obtained as

$$[\bar{K}]_e\{\bar{T}\}_e = \{\bar{R}\}_e \tag{5}$$

where the field variables, e.g., temperatures within each element and the associated temperature gradients, are approximated in the form

$$\bar{T} = [N(x_i)]_e \{\bar{T}\}_e \tag{6}$$

and

$$\left\{\frac{\partial \bar{T}}{\partial x_i}\right\}_e = [B]\{\bar{T}\}_e \tag{7}$$

where $[N(x_i)]_e$ and $[B]_e$ denote the temperature interpolation function and temperature-gradient interpolation function matrices, respectively. In Eq. (5), $[\bar{K}]_e$ has contributions from conduction and convection, and a convective-like stiffness matrix (the latter resulting from the transformed capacitance term), and $\{\bar{R}\}_e$ are contributions from element heat load vectors and initial conditions. Herein, we present the generalized element definitions for the heat-transfer cases considered.

$$[\bar{K}]_e = \int_{\Omega_e} [B]^T[k][B]\mathrm{d}\Omega_e + \int_{\Omega_e} \bar{\rho}\bar{c}[N]^T[N]\mathrm{d}\Omega_e + \int_{S_3} h[N]^T[N]\mathrm{d}S_3 \tag{8a}$$

$$\{\bar{R}\}_e = \int_{\Omega_e} \bar{Q}[N]^T\mathrm{d}\Omega_e + \int_{\Omega_e} \rho c T_i[N]^T\mathrm{d}\Omega_e + \int_{S_1} (\bar{q}\cdot\hat{n})[N]^T\mathrm{d}S_1 + \int_{S_2} \bar{q}_s[N]^T\mathrm{d}S_2 + \int_{S_3} h\bar{T}_\infty[N]^T\mathrm{d}S_3 \tag{8b}$$

The selection of the interpolation functions, which are functions of the space variables, is an important consideration for evaluating the matrices in Eqs. (8), and therein for accurately and effectively predicting the temperature distributions. Although not necessary, for certain class of problems, the selection of interpolation functions obtained via the general solution of the transformed differential equation is advantageous in that the corresponding basis functions generally yield more accurate results for the same element degree of freedom.

In general, the use of polynomials is very much permissible in the proposed formulations and the selection is arbitrary as in conventional formulations. For the numerical test cases presented, the interpolation functions used consisted of those derived via the general solution whenever feasible, as well as polynomial basis functions. For illustration purposes, consider the one-dimensional heat-transfer characteristics for a typical "thermal" model with the axial conduction heat-transfer mode or combined with internal heat generation, surface heat flux, or surface convection represented by

$$\rho c A \dot{T} = kAT_{,xx} + A\phi \tag{9}$$

where $\phi = 0, Q, q_s p, \ or \ -hp(T - T_\infty)$, respectively. Typical interpolation functions can be selected via the general solution of the transformed differential equations [applying the Laplace transform to Eq. (9)] and represented as

$$\bar{T} = [N_1 N_2 N_p]\{\bar{T}\}_e \tag{10}$$

where the following interpolation functions are proposed:

$$N_1 = \sinh[\lambda(l - x)]/\sinh[\lambda l] \tag{11a}$$

$$N_2 = \sinh[\lambda x]/\sinh[\lambda l] \tag{11b}$$

$$N_p = 1 - N_1 - N_2 \tag{11c}$$

and

$$\lambda = [(s/\alpha) + (s/\beta)]^{\frac{1}{2}}, \ \alpha = (k/\rho c), \ \beta = (skA/hp) \tag{12}$$

In fact, any appropriate interpolation functions may be used to approximate the temperature variation over an element via the proposed formulations. Again, for simplicity in illustrating the basic features, the interpolation functions for a two-noded linear element and a four-noded bilinear element are assumed in the form represented by

Two-noded linear element:

$$\bar{T} = [N_1 N_2]\{\bar{T}\}_e; \quad N_1 = -(\xi - 1)/2,\ N_2 = (\xi + 1)/2 \quad (13)$$

Four-noded bilinear element:

$$\bar{T} = [N_1 N_2 N_3 N_4]\{\bar{T}\}_e; \quad N_i = \tfrac{1}{4}(1 + \xi\xi_i)(1 + \eta\eta_i),\ i = 1,4 \quad (14)$$

After evaluating the element matrices [Eqs. (8)], the system equations are obtained next by summing the contributions of the individual elements, and are given as

$$[\bar{K}]\{\bar{T}\} = \{\bar{R}\} \quad (15a)$$

where

$$[\bar{K}] = \sum_{i=1}^{n} [\bar{K}]_e, \quad \{\bar{R}\} = \sum_{i=1}^{n} \{\bar{R}\}_e \quad (15b)$$

After having solved Eqs. (15), through an appropriate inversion technique, the solution response can now be readily obtained. Numerical inversion using transform methods was originally employed in applied mathematics.[4-6] Currently, several inversion techniques are available,[5,6] each having its pros and cons pertaining to accuracy and efficiency. For the examples presented, the technique of Durbin[7] was employed simply because of its high accuracy.

Applications

Structural Member with Axial Conduction Combined with Internal Heat/Surface Heat Flux/Surface Convection

A structural member of uniform cross section A and length L is subjected to axial conduction heat transfer combined with internal heat generation, surface heat flux, and surface convection. The numerical data for this test case is shown in Fig. 1. For this example, the member is modeled by: 1) two elements formulated via the selection of the interpolation functions from the general solution of the transformed equation so as to permit a node at the center, 2) 10 linear elements in the transform domain, and 3) 10 conventional linear elements using both consistent and lumped capacitance matrices and an implicit Crank-Nicolson algorithm. The comparative temperature his-

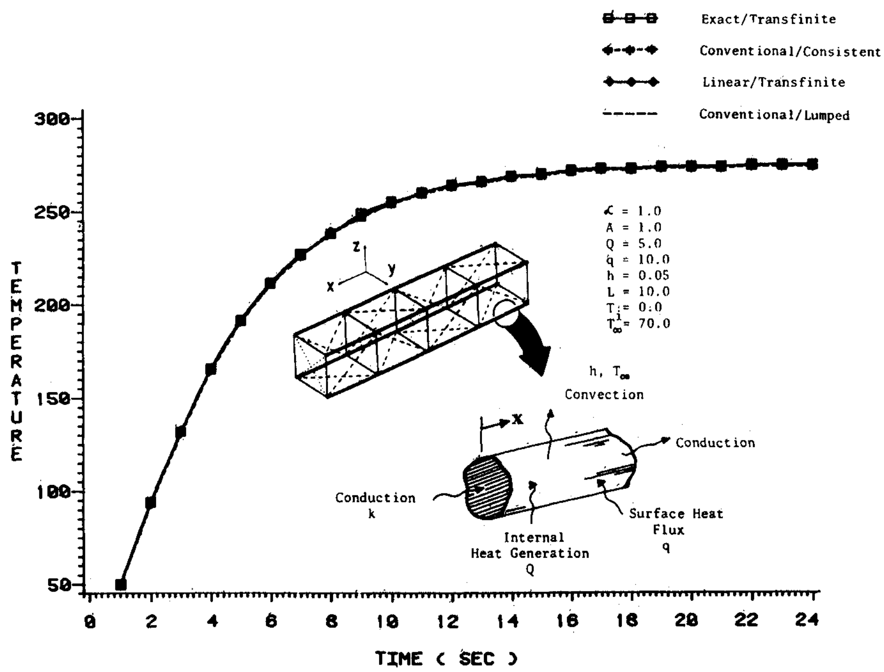


Fig. 1 Comparative thermal response of structural member with combined heat-transfer modes ($x = L/2$).

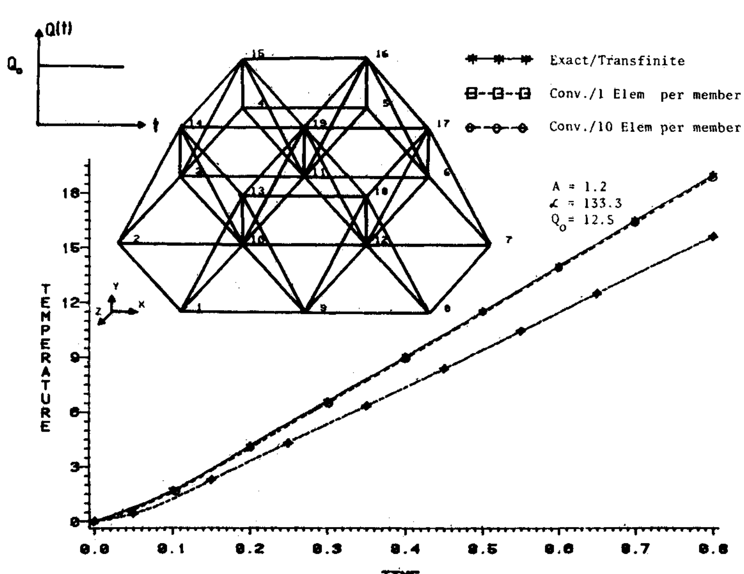
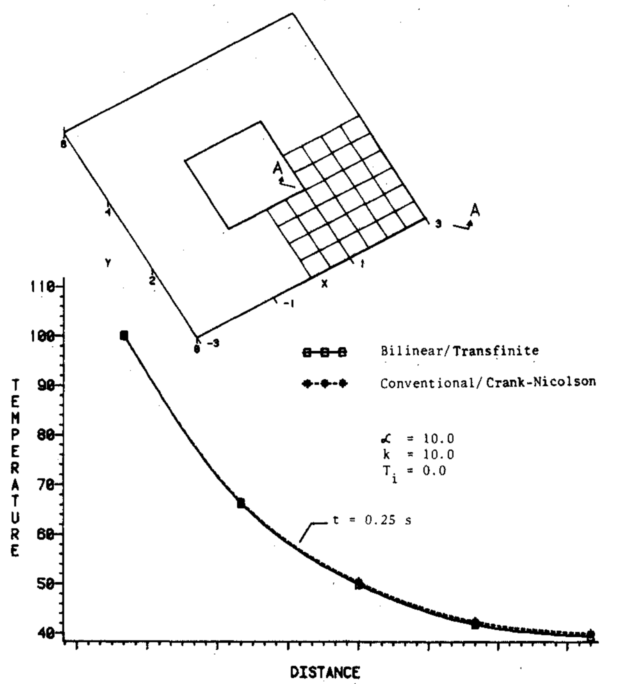


Fig. 2 Comparative thermal response for temperature history (node 12) for a three-dimensional tetrahedral space structure.

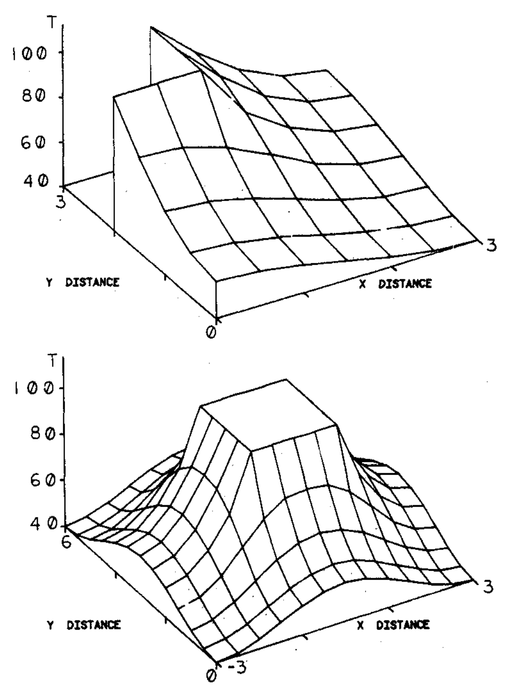
tories are shown in Fig. 1. The results are in excellent agreement and the accuracy of the results is within 1%.

Thermal Response of Tetrahedral Space Structure

A three-dimensional tetrahedral space structure (Fig. 2) has its top surface members subjected to a sudden constant surface heating. Tetrahedral-type truss structures are usually formed by structural members lying along the intersections of repeating tetrahedrons connected at their basis and vertices. The arrangement forms an upper and lower cover of triangular networks separated by a tetrahedron core. The data assumed for the computation of the thermal response are shown in Fig. 2. The space structure was modeled by 1) one element (formulated via the general solution of the transformed equation) per member, and 2) 10 conventional linear elements per member in order to capture the detailed temperature distributions. Figure 2 shows a typical comparative thermal response for the temperature history (node 12) together with the results of a conventional finite-element model that utilized one element and 10 linear elements per member, respectively. As shown in Fig. 2, the results of the refined conventional model approach the response predicted via the proposed formulation wherein, for these types of structures, the selection of the interpolation functions via the general solution of the transformed equation significantly reduces the model size.



a) Model and comparative distributions along A-A



b) Temperature distributions over one-quarter of the plate and the entire plate

Fig. 3 Comparative temperature distributions in a two-dimensional plate.

Two-Dimensional Transient Response in a Plate with a Hole

A two-dimensional square plate with a hole (Fig. 3) has its inside surface exposed to $T = 100$ and the outside edges are assumed insulated. The numerical data including the model are also shown in Fig. 3. Because of the symmetry considerations, only one-quarter of the plate is modeled. The plate was modeled by conventional finite elements using the Crank-Nicolson implicit algorithm and the transfinite-element formulations wherein four-noded bilinear elements were employed. The comparative temperature distributions along the plate diagonal at $t = 0.25$ s are shown in Fig. 3a. Figure 3b presents the temperature distributions over one-quarter of the plate and the entire plate at $t = 0.25$ s. The results are in excellent agreement, thereby demonstrating the fundamental capabilities of the proposed transfinite-element formulations.

Concluding Remarks

The development of an alternate methodology via transfinite-element formulations for thermal analysis of one- and two-dimensional structures is presented. The significance and characteristic features of the approach, including details of the solution scheme, are outlined. The applicability of the Laplace-transform-based finite-element methodology for transient thermal analysis demonstrated the excellent agreement of results with analytic solutions and/or conventional finite-element schemes. The transfinite-element approach provides features and an alternate methodology for thermal analysis of structures and is a viable alternative to existing techniques. Research is currently under way for further extensions. The approach offers potential for extension to general thermal analysis and for interdisciplinary research areas.

Acknowledgments

This research was supported in part by NASA Langley Research Center, Hampton, VA, and the Flight Dynamics Laboratory, Wright-Patterson Air Force Base, OH. The authors sincerely appreciate their continued encouragement and support.

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